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Application of the Finite Difference Method for Modelling Wave Propagation in a Gaseous Medium

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Abstract. The article contains a solution to the problem of wave propagation in a closed bounded region of the gaseous medium from two linear sources located along the walls of the considered region. To solve the problem, the finite-difference method is used in two simplest modifications: the Euler method and the predictor-corrector method. The boundary of the area is reflective. As a result, the iso-pole potential functions were obtained at different points in time.

1. Introduction

The simplest method for the numerical solution of differential equations is the finite-difference method, various implementations of which allow one to solve hyperbolic, parabolic, and elliptic equations. Wave processes in gaseous media are described by hyperbolic equations, which, unlike elliptic ones, require for their solution discretization not only of the spatial domain, but also the introduction of a time step, that is, space-time discretization. In the process of discretization, a transition occurs from an initially continuous differential problem to a finite system of algebraic equations with the desired values of the unknown function at the grid nodes.

The simplest method for solving the wave equations of the medium is the Euler method, which allows one to naturally solve the problem, relying on the values of the desired vectors on several time layers. In fact, the application of the Euler method consists in a sequential iterative calculation of the values of the desired functions of values at the nodes of the finite-difference space grid based on the previously calculated values of the unknown function, but on earlier layers. However, with the practical application of this method, a number of difficulties arise, one of which is the phenomenon of numerical instability, which consists in increasing the error of the solution with an increase in the number of the time layer.

A more progressive, but slightly more resource-intensive, representative of numerical methods, which allows one to obtain a numerical solution of the wave problem with attenuation, is the Euler method with recalculation ("predictor-corrector") [1]. Within the framework of which, at the first step, the value of the desired value of the function is calculated by the Euler method, and then it is refined.

2. Simulated features of wave phenomena

Elastic waves are waves associated with the oscillations of particles during mechanical deformation of an elastic medium. Under the conditions of the wave process, energy is transferred without the transfer of the substance itself. Sound waves arising in a gaseous medium under the influence of a disturbance are alternating areas of high and low air pressure, diverging from the source of the disturbance. Some specific inherent phenomena are characteristic of the wave process: reflection, refraction, diffraction, and interference of waves. Using the finite difference method, one can observe a number of such phenomena at different points in time. The article illustrates the reflection of a sound wave from the boundaries of the computational domain and the phenomenon of interference [2, 3], which consists in amplifying and attenuating the amplitudes of the wave at different points in the region at different points in time.

The theory of acoustic waves is engaged in linear and nonlinear acoustics. The wave process is called linear if the properties of the medium are independent of the wave intensity [4]. Linear waves do not affect the passage of other waves, that is, the waves propagate independently of each other without any distortion. The isopole given in the article illustrate the independence of two oscillatory processes arising in the medium as a result of the impact of two pulsing in-phase wave sources located on both sides of a closed computed rectangular region. The perturbation of the medium that appears at the initial moment of time sequentially moves from two linear sources combined with the region's boundaries and captures more and more new areas of the computational domain until two "humps" disturbances occur in the center of the region, then the humps pass through each other, mutually reinforcing and weakening each other, the phenomenon of interference occurs. The article deals with the propagation of linear waves. A much more complicated phenomenon is the propagation of nonlinear waves, while in the medium under the influence of a wave, the properties of the medium itself change, and this, in turn, changes the properties of the wave. In particular, the work [4] is devoted to the problem of the propagation of nonlinear waves.

3. Formulation of the problem

As is known [2,3], in the case of wave propagation in liquids and gases, only longitudinal waves are observed, shear waves are absent. This is due to the fact that solids, in contrast to liquids and gases, possess elasticity not only in volume but also in shape. In the process of wave propagation in a liquid or gas, a change in pressure and density of the medium occurs at different points in time; we denote $p = p_0 + p_a$-instantaneous pressure of individual points of the medium, where p_0 is pressure in an unperturbed medium, pa is the pressure from the development of the wave process. $\rho = \rho_0 + \rho_a$, where ρ_0 is the density of an unperturbed medium, ρ_a is the density of the medium, taking into account the development of the wave process. In the framework of linear acoustics, it is assumed $p_a << p_0$, $\rho_a << \rho_0$.

The basic equations of the acoustic field [5] in the absence of sources is the equation of motion of particles of a continuous medium (Newton's equation):

$$\frac{dV}{dt} + \frac{1}{\rho_0} \cdot grad\left(p_a\right) = 0, \tag{1}$$

where V is the velocity vector of individual points of the medium, is time, ρ_0 is the density of the medium in the unperturbed state, p_a is the change (perturbation) pressure associated with the wave process.

The continuity equation expressing the law of conservation of mass of a substance:

$$\frac{d\rho_a}{dt} + \rho_0 \cdot div\left(V\right) = 0, \tag{2}$$

where ρ_a is the density change associated with the development of the wave process, ρ_0 is the density of the medium in the unperturbed state, and V is the velocity vector.

The equation of state [5], expressing the relationship between density and pressure:

$$p_a = K\frac{\rho_a}{\rho_0}, \tag{3}$$

where K is the module of bulk elasticity.

Omitting the intermediate calculations and assuming that the oscillatory process in a liquid or gas is very small oscillations $p_a << p_0$, $\rho_a << \rho_0$, then the velocity vector can be taken proportional to the gradient of some scalar function ϕ (x, y) depending on the spatial coordinates of points areas [5]:

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$$V = \frac{1}{\rho} grad\left(\varphi\left(x, y, t\right)\right), \tag{4}$$

We present the wave equation for the potential, which describes the wave process in the case of considering an ideal liquid or gas:

$$\frac{d^2\varphi\left(x, y, t\right)}{dt^2} + c^2\Delta\left(\varphi\left(x, y, t\right)\right), \tag{5}$$

where c is the wave velocity in the liquid or gas.

Further, on the basis of the calculated potential, at each time instant it is possible to calculate the components of the velocity vector V and the values of pressure and density.

In this article, we consider a square region whose borders are located along the corresponding coordinate axes. Figure 1 shows the appearance of the region and the location of two sources of waves located symmetrically relative to the axis of symmetry X. The region's points located along the Y axis are free from the sources. This arrangement of sources allows to obtain the expected interference pattern at the initial stages of the wave process (in particular, a twofold increase in the amplitude of the unknown function in the center of the region). The amplitudes of the oscillations of the sources are the same, the phases of the oscillations coincide.

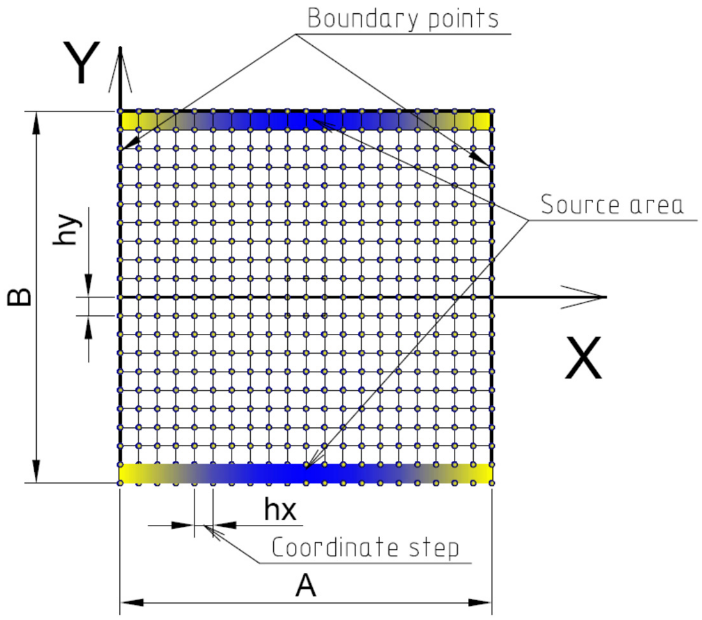

Figure 1. The calculated region in the Cartesian coordinate system, indicated the region of the location of the sources of oscillations

4. Boundary and initial conditions

The article discusses the simplest type of boundary condition, which consists in the equality to zero of both the normal and tangential components of the velocities at the points of contact with the boundary of the region. Since the sources act impulsively, only in the first few moments of the wave process development, then at the subsequent stages of wave process development, the boundary points of the region adjacent to linear sources also become part of the boundary of the region with equal normal and tangential velocity components.

As initial conditions, we assume that the medium before the wave process was in an unperturbed state, that is, both the potential of the field ϕ (x, y) = 0 and diff (ϕ (x, y), t) = 0. For a short time of ~ 0.27 sec, the sources affect the fragment under consideration (along the walls parallel to the X axis),

causing oscillations. Further, the process of wave propagation occurs already without obvious influence from the sources.
The condition at the border to a greater extent describes the behaviour of the medium, which “sticks” to the walls of the boundary of the region, forming a certain “near-wall” layer of the medium, where the motion of the medium is completely inhibited due to the phenomenon of sticking to the walls.

5. The Euler method and the predictor-corrector method

Euler's method-- representative of numerical methods for solving the Cauchy problem [6,7]. In relation to the problem under consideration, the main relations of the method will be as follows:

$$\varphi_{i,j,k+2} = 2\varphi_{i,j,k+1} - \varphi_{i,j,k} + c^2 h_t^2 \left\{ \frac{\varphi_{i+2,j,k} - 2\varphi_{i+1,j,k} + \varphi_{i,j,k}}{h_x^2} + \frac{\varphi_{i,j+2,k} - 2\varphi_{i,j+1,k} + \varphi_{i,j,k}}{h_y^2} \right\}, \tag{6}$$

where:
i, j-indices characterizing the spatial location of the point;
the k-index characterizing the time layer for direct methods, the calculation on subsequent time layers obviously depend only on the values on the previous time layers;
hx, hy-steps by spatial coordinate;
ht- time step;
c is the wave velocity in a liquid or gas.
When applying this simple algorithm, it is necessary to monitor compliance with the Courant condition, which imposes restrictions on the time integration step.
In addition to this simple algorithm, the use of the Euler method with recalculation (the predictor-corrector method) [1, 7] is somewhat more complicated. When it is implemented in the first step, the values are calculated by the Euler method, and then the adjustment “adjustment” takes place. Finally, the sequence of formulas of the predictor-corrector method applied to the problem in question will be:

$$\begin{aligned}
&\varphi 1_{i,j,k+1} = \varphi_{i,j,k} + h_t \alpha 1_{i,j,k} \\
&\alpha 1_{i,j,k+1} = \alpha 1_{i,j,k} + c^2 h_t \left\{ \frac{\varphi_{i+2,j,k} - 2\varphi_{i+1,j,k} + \varphi_{i,j,k}}{h_x^2} + \frac{\varphi_{i,j+2,k} - 2\varphi_{i,j+1,k} + \varphi_{i,j,k}}{h_y^2} \right\} \\
&\varphi_{i,j,k+1} = \varphi_{i,j,k} + 0.5 \cdot h_t \left(\alpha 1_{i,j,k} + \alpha 1_{i,j,k} \right) \\
&\alpha_{i,j,k+1} = \alpha 1_{i,j,k} + 0.5 \cdot c^2 h_t \left\{ \frac{\varphi_{i+2,j,k} - 2\varphi_{i+1,j,k} + \varphi_{i,j,k}}{h_x^2} + \frac{\varphi_{i,j+2,k} - 2\varphi_{i,j+1,k} + \varphi_{i,j,k}}{h_y^2} \right\} + \\
&+0.5 \cdot c^2 h_t \left\{ \frac{\varphi 1_{i+2,j,k} - 2\varphi 1_{i+1,j,k} + \varphi 1_{i,j,k}}{h_x^2} + \frac{\varphi 1_{i,j+2,k} - 2\varphi 1_{i,j+1,k} + \varphi 1_{i,j,k}}{h_y^2} \right\},
\end{aligned} \tag{7}$$

where the introduced value α (x, y, t) = diff (ϕ (x, y, t), t) is introduced to reduce the original equation of the second order to the system of equations of the first order and the subsequent application to it of the standard procedure of the predictor-corrector method .

In a number of courses of numerical methods [1], it is proved that the Euler method is a first-order accuracy method, while the Euler method with recalculation (the predictor-corrector method has already a second order of accuracy).

6. Results and discussions

As a solution, we obtained the isopolis of the potential of the acoustic field ϕ (x, y, t) at different times at all points of the considered region. From the graphs obtained it can be seen that the started oscillatory process reaches the walls of the area under consideration, is reflected from them, undergoes mutual amplification and a decrease at certain points in the area under consideration. Further, on the basis of the potential isopoles obtained, both the isopole components of the velocity vector and the isopole of pressure and density at all points of the considered region at any time can be easily obtained. The calculation formulas of the Euler method and the Euler method with recalculation (the predictor-corrector method) given in the previous paragraph are not quite identical in terms of the cost of computer resources. In particular, the Euler method requires storing information about the desired vector of potential only on three time layers: the current one and the two preceding ones, while the predictor-corrector method makes it necessary to store at each step of the computational process not only the values of the potential ϕ (x, y, t) and its initial approximation ϕ1 (x, y, t), but also α (x, y, t) and α1 (x, y, t) values equal to the derivatives of the potential with respect to time for each of the points considered areas, but already on two three temporary layers, that is, the predictor-corrector method is more demanding of memory, but having at the same time twice the accuracy of the solution. The accuracy of solving wave problems has the greatest value for obtaining solutions on time layers that are significantly removed from the initial starting point in time, this is due to the gradual accumulation of errors in the process of implementing the calculations. However, increasing the requirements for the data stored at each step of the data improves the accuracy of the solution.

The following numerical values of the steps in the coordinates were used in the calculations: hx, hy = 0.1 m., The area size is 100x100 points, the time step is ht = 0.01 sec. The wave propagation velocity is c = 343 m / s.

Figures 2-5 show some characteristic isopoles and graphs of solutions at different points in time, obtained using the Euler method and the predictor-corrector method. From the obtained solution graphs it can be seen that over time, the solution obtained by the Euler method practically does not fade, while the use of the predictor-corrector method is self-extinguishing. Special attention is given to the presence of a larger number of "folds" in the isopoles obtained by the Eulera method, while the solution obtained by the predictor-corrector method does not contain such artifacts, that is, it describes more accurately the real oscillatory process in the medium including during periods time, given from the time of the beginning of observations.

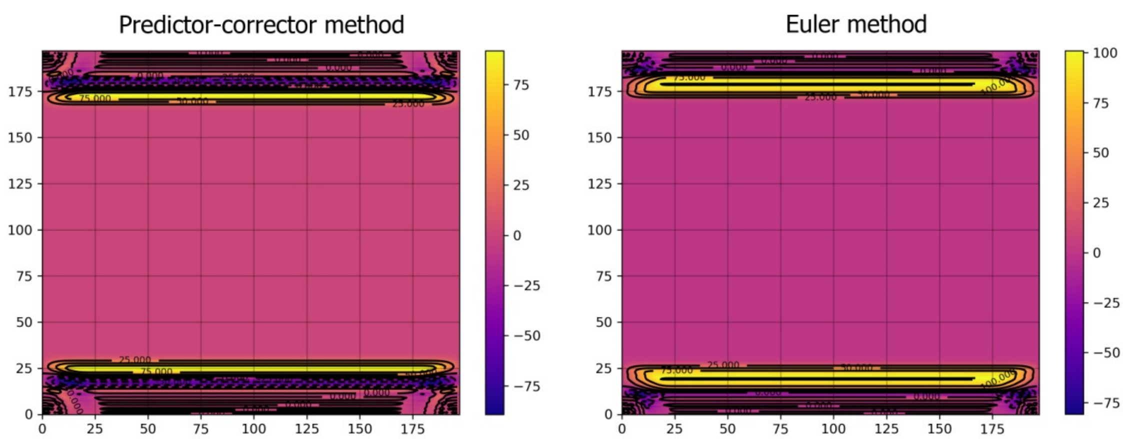

Figure 2. Isopole of the potential function ϕ (x, y, t) with t = 0.015 s
(waves emanating from two sources have not yet met in the middle of the computational domain)

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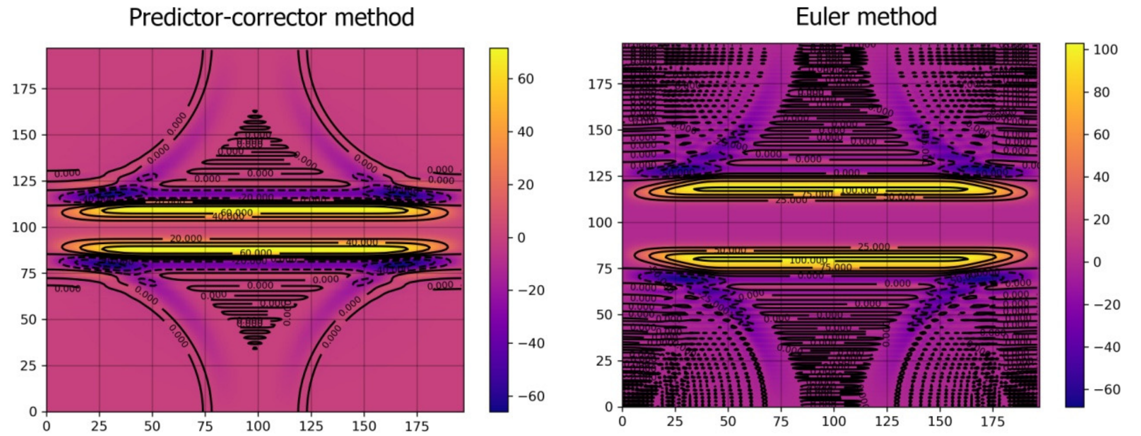

Figure 3. Isopole of the potential function ϕ (x, y, t) with t = 0.025 s
(the moment preceding the meeting of two waves in the center of the region)

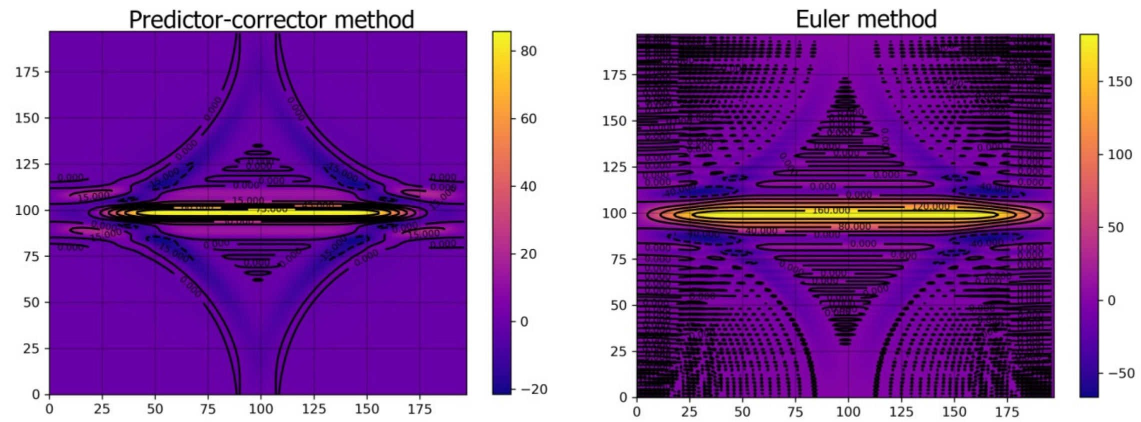

Figure 4. Isopole of the potential function ϕ (x, y, t) with t = 0.030 s
(the moment of meeting two waves in the center of the region)

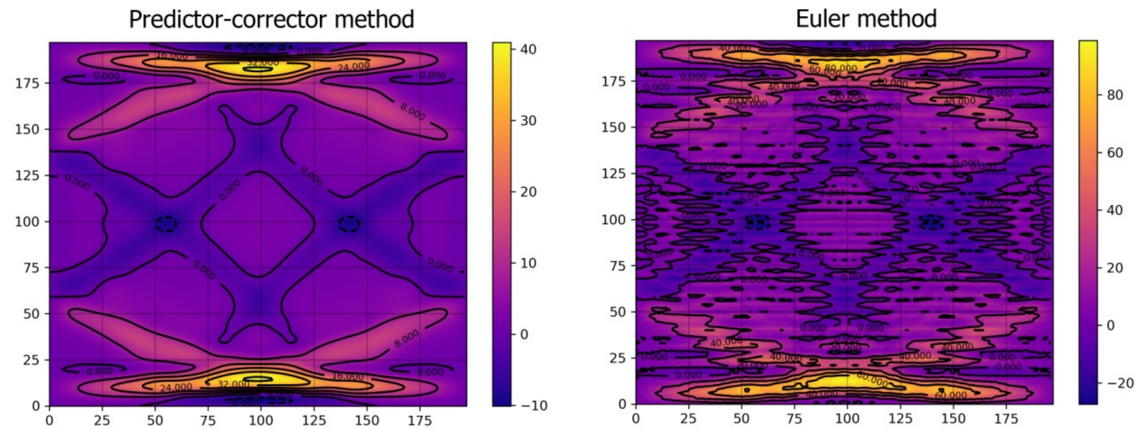

Figure 5. Isopole of the potential function ϕ (x, y, t) with t = 0.030 s
(wave interference, the Euler solution does not fade with time)

7. Conclusions

An algorithm for constructing a numerical solution of the problem of wave propagation in a gaseous environment in a limited area using the Python programming language and the graphic library matplotlib has been implemented.

The application of the step-by-step direct Euler iterative method and the predictor-corrector type method is considered. The obtained solutions illustrate the formation of reflected waves when the wave reaches the boundary of the region, as well as the appearance of some non-smoothness of the solution when using the Euler method, which is explained, among other things, by the first order of accuracy of the Euler method.

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